Turbulent transport of heat and momentum from an infinite rough plane

By T. H. ELLISON
Sidegarth, Heswall, Cheshire

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SUMMARY

In the first part of the paper the dimensional laws governing the processes of heat and momentum transport from an infinite rough plane are assembled and their consequences set out. In the second part, the detailed equations for the turbulent energy, the mean square temperature fluctuation and the covariance of temperature and vertical velocity are used, together with some speculative assumptions concerning the dissipative action of the turbulence, to derive a series of relations between the turbulent intensities and the *Austausch* coefficients. One of these relations indicates that the flux form of the Richardson number cannot exceed a critical value which is about 0·15. It follows that in highly stable conditions the buoyancy forces have little direct effect on the turbulent energy balance, their action being primarily to cause a reduction in the scale of the motion and some change in its structure.

1. INTRODUCTION

The theory of turbulent flow near a heated or cooled horizontal rough surface is notoriously difficult, but in the past few years some progress has been made. This has largely been a result of an increased understanding of the role of the various physical quantities in the problem and the more precise formulation of a theoretical model, which have enabled certain relationships to be put forward on dimensional grounds. These restrict the possible forms of certain functions (e.g. the velocity profiles) and show that some of the empirical formulae hitherto used to describe them (e.g. Deacon's (1949) law) cannot be correct. It is the aim of the first part of this paper (§ 2 to § 5) to collect together these dimensional arguments and display the correct form of the physical laws. Nearly all of the steps in this demonstration have been described by earlier writers or are implicit in their work; but individual papers have presented only fragments of the argument and these have frequently been associated with assumptions which are very unlikely to be correct (e.g. that the *Austausch* coefficients for heat and momentum are equal). Thus, since it is only after the dimensional arguments have been completed that it is profitable to examine the mechanism of the flow in detail, the author hopes that any readers who are familiar with these arguments will excuse their repetition.

The character of the second part of the paper (§ 6) is different. In it an attempt is made to learn more about the physical processes controlling the transfer of heat and momentum from the equations of motion. In order to do this, the detailed equations for the turbulent energy, the mean square density fluctuation and the covariance between density and vertical velocity are written down. These alone are not sufficient to solve the problem, and to make further progress it is found necessary to resort to some speculation and guess-work; thus the results that are obtained, although based on assumptions which the author finds plausible, cannot be regarded as having any established validity. These assumptions concern the dissipative action of the turbulence on the turbulent energy, the density fluctuations and the covariance of the density and vertical velocity. It is suggested that the rates at which the turbulence would begin to destroy these quantities, if the simple processes continually creating them were to cease to operate, vary with the stability in such a way that their ratios remain constant. It is shown on the basis of this assertion that the ratio of the *Austausch* coefficient for heat to that for momentum approaches zero when the flux form of the Richardson number approaches a critical value which is considerably less than unity. It follows that in very stable conditions it is not through the energy balance that the buoyancy forces exercise their great influence on the properties of the turbulence. The intensity of the density fluctuations is shown to be an important quantity, and that further investigation of the way in which it is controlled is desirable.

2. The theoretical model

The boundary layers of meteorology commonly differ from those of aerodynamics in that their thickness is great and the region that is of primary interest and accesible for measurement is relatively small and close to the surface. The actual thickness and possible growth of these layers is often assumed to be of little importance, and it is customary to treat them theoretically as if the properties near the surface were independent of the state of the flow at great heights. Moreover, the turbulent shear-stress and heat flux are considered to be independent of height. This corresponds closely to the actual conditions near the ground so far as the shear-stress is concerned, but unfortunately the complex effects of radiation do in practice lead to some change with height in the turbulent heat flux. These are of dominant importance in stable conditions when the turbulent *Austausch* coefficient is very small, and possibly also very near the surface. In addition, it is seldom that the ground or sea is sufficiently uniform horizontally for advection to be negligible and a steady and homogeneous state to be established. For the present, theory must ignore these complications and consider a steady state in which the turbulent fluxes are both independent of height.

It is now possible to specify a theoretical model. We consider an infinite uniform rough plane ($z = 0$) which supports a fluid of great depth.

A stress ρu_*^2 is applied to the plane tangentially causing it to move in the negative x-direction, and heat is supplied to (or extracted from) the plane at a constant rate. We take our frame of reference to be fixed relative to the plane and confine our attention to the steady state which the flow in the vicinity of the plane is supposed to attain ultimately. In this model the mean velocity of the fluid (relative to the plane) increases indefinitely with height in an unknown way; and because of this, the indirect and rather unnatural specification of the flow is needed. The unproved assumption that the flow near the surface attains a steady state is important as it implies that the flow there is not influenced by that at great heights. If it is false and the flow near the surface does depend on the thickness of the boundary layer, the great complexity of the meteorological problem would indicate little prospect of deriving any theory at present.

We shall consider the fluid to be incompressible, but the known equivalence (subject to certain restrictions) between the motion of an incompressible fluid and that of a gas, enables the results to be applied to the atmosphere as required. Moreover, we shall assume that the fractional changes in density are so small that their effect on the inertia of the fluid can be neglected, so that they need only be taken into account in the gravitational terms. In other words, the fluid is treated as having *uniform density but variable weight*, as is commonly done in problems of this type.

3. The role of z_0

In conditions of neutral stability* (that is, when there is no heat flux) it is known that the mean velocity U at height z is given by the equation

$$\frac{dU}{dz} = \frac{u_*}{kz}, \tag{1}$$

where k is a constant with an observed value of about 0·4. The integral of (1) is

$$\frac{kU}{u_*} = \log(z/z_0), \tag{2}$$

where z_0 is a length characterizing the roughness of the surface; z_0 enters (2) only through the boundary condition determining the constant of integration of (1), and so a change in z_0 merely adds a constant velocity to the flow. It would seem reasonable to suppose that such a change would not affect the turbulence in any way (except at heights of the order of z_0). Even in stable and unstable conditions, (1) still holds near the surface and we are able to state quite generally that *the only effect of a change in the value of z_0 is to superpose a uniform translation on the whole flow without modifying its internal mechanism.*

* The word ' stability ' is used in this paper to indicate the static stability of an inviscid fluid with the same vertical distribution of potential density as that in the flow considered; it does not refer to the growth of small disturbances in the moving fluid.

This statement is so obvious and elementary that at first sight it seems strange that it has received little attention from meteorologists. The explanation of this probably lies in uncertainty concerning the assertion that the flow near the surface is independent of that at great heights. When the theory of the whole depth of the boundary layer (Ekman spiral) is considered, even in its simplest form (see Ellison 1956), it is found that z_0 is involved in determining its thickness. Thus, unless the assertion is correct, z_0 is likely to affect the whole structure of the flow.

4. Dimensional arguments

The statement of the last section has some immediate and far-reaching consequences. It implies that the only variables determining the internal structure of the flow are $g\overline{\rho' w}/\bar{\rho}$ (where ρ' is the density fluctuation and $\bar{\rho}$ the mean density), ν (the molecular kinematic viscosity), κ (the thermometric conductivity), u_* and z. At the very high Reynolds numbers of the atmosphere the molecular quantities may be omitted where they occur separately, but it is debatable whether or not the Prandtl number ν/κ ought to be considered in relation to heat transfer (Batchelor & Townsend 1956). It is omitted here, but since it is a dimensionless constant the omission has little importance at this stage. Hence, in neutral conditions all dimensionless groups such as $\overline{u^2}/u_*^2$, $\overline{v^2}/u_*^2$, $\overline{w^2}/u_*^2$, $(z/u_*)dU/dz$, $z^2\overline{(\partial p/\partial x)^2}/u_*^4$, etc., (where u, v and w are the components of the velocity fluctuation and p is the pressure fluctuation) must be constant and independent of height. Covariances such as $\overline{u(x_1, y_1, z_1)u(x_2, y_2, z_2)}$ must be functions of only $(x_1-x_2)/z_1$, $(y_1-y_2)/z_1$, and $(z_1-z_2)/z_1$*.

When there is a heat flux, an additional length $L = \bar{\rho}u_*^3/g\overline{\rho' w}$ is available and all the dimensionless groups will depend on the dimensionless height variable, $Z = z/L$ (Obukhov 1946; Charnock 1956). Thus, using the hypothesis that even in non-neutral conditions the effect of a change in z_0 is merely to superpose a uniform translation on the whole flow, we have

$$\frac{1}{u_*}\frac{dU}{d(\log|Z|)} = f'(\log|Z|, \operatorname{sgn} Z). \tag{3}$$

For small values of Z the buoyancy forces are negligible and f' must approach the constant value $1/k$ which it takes in neutral conditions. Thus, the integral of (3) is

$$\frac{kU}{u_*} + \log|S_n| = kf(\log|Z|, \operatorname{sgn} Z), \tag{4}$$

where S_n is Businger's (1955) stability index defined by

$$S_n = z_0/L = Zz_0/z, \tag{5}$$

* These variables can, of course, be combined in various ways, e.g. to give $(x_1-x_2)/(z_1+z_2)$, etc. The choice of the most convenient form is a matter of some delicacy.

and $f(\log|Z|, \operatorname{sgn} Z)$ tends to $\log|Z|$ for small Z. It will be seen from (4) that the velocity profile can take only two forms, one for stable and one for unstable condition; the logarithmic law is the limit of these for small heights.

The determination of the function $f(\log|Z|, \operatorname{sgn} Z)$ in (4) is, of course, a major task for theory, but not one that can yet be accomplished. However physical reasoning can take us a little farther. In 1920 Richardson showed that the quantity which is now known as the flux form of the Richardson number (Deacon 1955) (and denoted here by Rf) is equal to the ratio of the rate at which buoyancy forces extract energy from the turbulence to the rate at which it is supplied by the shear-stress. This result is a direct consequence of the energy equation for the turbulence, (13). Rf may be expressed in the following equivalent ways:

$$Rf = Ri\,\frac{K_H}{K_M} = -\frac{K_H g\, d\bar{\rho}/dz}{K_M (dU/dz)^2} = \frac{g\overline{\rho' w}}{\bar{\rho} u_*^2\, dU/dz} = \frac{g\overline{\rho' w} K_M^* z}{\bar{\rho} u_*^3} = Z K_M^*, \quad (6)$$

where $$K_M^* = \frac{K_M}{u_* z} = \frac{u_*}{z(dU/dz)} \quad \text{and} \quad K_H = -\frac{\overline{\rho' w}}{d\bar{\rho}/dz}.$$

Richardson's arguments make it clear that Rf cannot exceed unity over any substantial depth of fluid, so it must have a maximum value (critical value) not greater than this. It will be suggested shortly that Rf_{crit} is in fact considerably less than one, but some deductions can be made from its mere existence. Rf and K_M^* are functions of Z, and physical intuition urges that they are monotonic functions. In that case, Rf must approach its critical value when Z becomes large and positive. We have in the limit:

$$Rf_{\text{crit}} = Ri K_H/K_M = g\overline{\rho' w} K_M/\bar{\rho} u_*^4. \quad (7)$$

Hence in very stable conditions K_M is independent of height and determined by the heat and momentum fluxes. Linear velocity profiles such as are predicted by (7) have been observed in the atmosphere (e.g. by Rider & Robinson 1951), but the agreement is probably fortuitous since it is likely that in the conditions of the measurements the temperature profiles were controlled by radiation and that the turbulent heat flux was not independent of height.

If Ri, as well as Rf, has a maximum value, (7) determines K_H in terms of it; but there is no reason to think that this is the case. It seems more likely (especially in view of (20)) that turbulence can be maintained at large values of Ri. Proudman (1953), following earlier work of Sir Geoffrey Taylor, has analysed measurements of currents and salinity in the *Kattegat* and found appreciable transport of momentum by turbulence associated with values of Ri up to 10. As Ri increases, the ratio K_H/K_M must decrease in proportion; so, in our model, K_H decreases indefinitely with height and is not determined uniquely by the fluxes alone in the same way as K_M. In order to find K_H, and hence the density profile, it is necessary to know how the limit is approached. That is, one must know how K_H/K_M approaches zero as Rf approaches Rf_{crit}, and how Rf varies with Z.

At the other extreme, in very unstable conditions (large negative Z), a state of free convection may be assumed to exist, so that K_H is independent of u_*. It follows that the quantity H^* defined by,

$$H^{*2} = -\frac{\overline{\rho' w^2 \rho}}{g z^4 (d\bar{\rho}/dz)^3}, \tag{8}$$

is constant. Measurements made by workers in Australia (Taylor 1956) suggest that it has a value of about 0·8. An alternative expression for H^*, which brings out the dependence of K_H^* on Z in these conditions, is

$$H^{*2} = -K_H^{*3} Z^{-1}. \tag{9}$$

Any sort of mixing length theory would seem to suggest that K_M is also independent of u_* in free convection, and this is generally assumed to be the case. It follows that K_H/K_M must approach a constant value for large negative Z.

Our knowledge of K_M^* as a function of Rf can now be summarized as follows. When $Rf = 0$, $K_M^* = k$ (see (1)); when Rf approaches Rf_{crit} at large positive Z, K_M^* vanishes (see (7)); when Rf is large and negative, K_M^* is proportional to $Rf^{1/4}$ (see (9) and (6)). Hence the well known and useful semi-empirical formula of Holzman (1943),

$$(K_M^*/k)^2 = 1 - sRi, \tag{10}$$

where s is a constant, cannot be correct except for very small values of Ri. However, it is very convenient to have a semi-empirical formula like this, and so the following may be suggested:

$$(K_M^*/k)^4 = 1 - Rf/Rf_{\text{crit}}. \tag{11}$$

Note that the right hand side of (11) contains Rf in place of Ri. It can be written as a relation between K_M^* and Z, since by (6) we have

$$(K_M^*/k)^4 = 1 - ZK_M^*/Rf_{\text{crit}}.$$

(11) has no theoretical basis beyond having the correct form in limiting conditions and presumably a more elaborate formula with more empirical constants will be needed as soon as precise observations become available.

In the case of free convection (11) implies

$$K_H/K_M = Rf_{\text{crit}}^{1/3} H^{*2/3} k^{-4/3}. \tag{12}$$

When one inserts the numerical values already mentioned for K^* and k and the value 0·15 for Rf_{crit} as is suggested below, this gives $K_H/K_M = 1{\cdot}6$, which seems quite reasonable.

5. The (log + linear)-law

Presumably K_M^* can be expanded in a power series in Z:

$$K_M^*/k = 1 + a_1 Z + a_2 Z^2 + \dots,$$

so

$$kU/u_* = \log(z/z_0) - a_1 Z + \dots\,. \tag{13}$$

Businger (1955) has shown from an analysis of Rider's (1954) velocity profiles that a_1 is about 0·8, while Monin (undated) gives $a_1 = 0{\cdot}6$.

The semi-empirical equation (11) yields $a_1 = k/4Rf_{\text{crit}}$, which implies that $Rf_{\text{crit}} = 0{\cdot}14$ if $a_1 = 0{\cdot}7$. This value has no more authority that (11) itself, but it does strongly indicate that Rf_{crit} is small. It also turns out that this value is virtually identical with that suggested by a quite separate argument in the next section.

6. The transport mechanism

In this section an attempt is made to approach the subject from a new angle and to derive useful information from the equations of motions together with some relatively weak and general assumptions. The suggestion already made that Rf has a small critical value is confirmed, and it is shown that the assumptions lead to a reasonably consistent physical picture of the processes governing the flow. It is not claimed that the validity of the assumptions is established; but, whether they are correct or not, the theory does indicate subjects which are likely to repay further investigation.

From the Navier–Stokes equations, the equation of continuity and the equation of heat conduction, it is elementary to deduce the following equations for the mean square density fluctuation, the turbulent energy and the density flux:

$$\frac{1}{2}\frac{\partial\overline{\rho'^2}}{\partial t} + \frac{1}{2}\frac{\partial\overline{w'^2\rho}}{\partial z} + \overline{w\rho'}\frac{d\bar{\rho}}{dz} - \kappa\overline{\rho'\nabla^2\rho'} = 0, \tag{14}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{u^2}+\overline{v^2}+\overline{w^2}) + \frac{1}{2}\frac{\partial}{\partial z}(\overline{u^2w}+\overline{v^2w}+\overline{w^3}) + \overline{uw}\frac{dU}{dz} + \\ + \frac{1}{\bar{\rho}}\frac{\partial\overline{wp'}}{\partial z} + \frac{\overline{w\rho'}g}{\bar{\rho}} - \nu(\overline{u\nabla^2u}+\overline{v\nabla^2v}+\overline{w\nabla^2w}) = 0, \tag{15}$$

$$\frac{\partial\overline{w\rho'}}{\partial t} + \frac{\partial\overline{w^2\rho'}}{\partial z} + \overline{w^2}\frac{d\rho}{dz} + \frac{1}{\bar{\rho}}\overline{\rho'\frac{\partial p}{\partial z}} + \frac{\overline{\rho'^2}g}{\bar{\rho}} - \kappa\overline{w\nabla^2\rho'} - \nu\overline{\rho'\nabla^2w} = 0. \tag{16}$$

In our model conditions are steady and so the first terms in these equations vanish; they have been written down merely to aid understanding. The 'diffusion' terms like $\partial(\overline{w\rho'^2})/\partial z$, etc., are probably always negligible since $\overline{\rho'^2}$, $\overline{u^2}$, etc., vary only slowly, if at all, with height. The 'pressure-flow' term $\partial(\overline{pw})/\partial z$ in (15) is zero in neutral conditions, and there seems no reason to think that it ever becomes significant. There remain the dissipation terms, the term $\overline{\rho'\partial p/\partial z}$ in (16), and the simple terms of obvious interpretation among which we wish to discover new relations.

Now it is well known that at high Reynolds numbers the rates of dissipation are very largely determined by typical length and velocity scales of the turbulence and not by molecular quantities such as viscosity. Let us therefore formally introduce decay times T_1, T_2, and T_3, for $\overline{\rho'^2}$, $\overline{u^2}+\overline{v^2}+\overline{w^2}$, and $\overline{w\rho'}$, such that, in the (imagined) absence of the producing terms, the mixing action of the turbulence would begin to destroy these quantities at rates equal to $1/T_1$, etc. In other words, let us *define* T_1, T_2 and T_3 by the following equations:

$$\frac{\overline{\rho'^2}}{2T_1} + \overline{w\rho'}\frac{d\bar{\rho}}{dz} = 0, \tag{17}$$

$$\frac{\overline{u^2}+\overline{v^2}+\overline{w^2}}{2T_2} + \overline{uw}\frac{dU}{dz} + \overline{w\rho'}g = 0, \tag{18}$$

$$\frac{\overline{w\rho'}}{T_3} + \overline{w^2}\frac{d\bar{\rho}}{dz} + \frac{\overline{\rho'^2}g}{\bar{\rho}} = 0. \tag{19}$$

Since the mixing action of the turbulence tends to destroy the correlation between w and ρ', as well as $\overline{w^2}$ and $\overline{\rho'^2}$, T_3 may be expected to be substantially smaller than T_1 and T_2, which may well be roughly equal. While T_1, T_2 and T_3 certainly vary with stability, we may reasonably guess that they are all dominantly determined by the same typical length and velocity scales and that the relation between the processes responsible for the destruction of the turbulent energy, the density fluctuations, and the density flux, change only little with stability. If so, T_1, T_2 and T_3 must remain in roughly fixed ratios. Such an assertion is admittedly speculative, but seems to lead to a consistent physical theory.

In (19), the term $\overline{\rho'\partial p/\partial z}$ has been incorporated in $\overline{w\rho'}/T_3$. This is plausible if the term is considered to represent the drag on a rising blob of hot fluid in unstable conditions; in stable conditions the term turns out to be negligible and so the interpretation is unimportant.

It may be noted at once that (17) alone implies that K_H is always positive and that no selective action of the buoyancy forces, such as was once suggested by Priestley & Swinbank (1947), can cause a transfer of density against the gradient in stable conditions.

After a little algebra, the equations (17) to (19) can be rearranged to give K_H/K_M, T_2 (say), and $\overline{\rho'^2}$ in terms of the known fluxes, the ratios of T_1, T_2 and T_3, and the components of the turbulent energy. Let us first examine the equation for K_H/K_M:

$$\frac{K_H}{K_M} = \frac{(\overline{u^2}+\overline{v^2}+\overline{w^2})\overline{w^2}[1-Rf(1+T_1(\overline{u^2}+\overline{v^2}+\overline{w^2})/T_2\overline{w^2})]}{2u_*^4(T_2/T_3)(1-Rf)^2}, \tag{20}$$

which is the basic result of this part of the paper. There can be little doubt that for small positive values of Rf the term in square brackets is the dominant one. If we take $T_1/T_2 = 1$ and $(\overline{u^2}+\overline{v^2}+\overline{w^2})/\overline{w^2}$ to have the same value as in neutral conditions, which is known to be roughly 5·5, we see that K_H/K_M falls to zero when $Rf = 0{\cdot}15$. This small critical value of Rf is most remarkable since it implies that in stable conditions the buoyancy forces do not have any great effect on the energy balance. In these circumstances it is difficult to believe that the assumption that $(\overline{u^2}+\overline{v^2}+\overline{w^2})/\overline{w^2}$ has the same value as in neutral conditions is far wrong; but, in any case, one would expect that in stable conditions horizontal motions would be favoured at the expense of vertical ones, thereby causing the quantity to increase rather than decrease. Such a change would lead to an even smaller value of Rf_{crit}. It is, of course, conceivable that T_1/T_2 decreases in stable conditions, but there is no obvious reason why this should be expected, and it would require a very drastic change to affect the general conclusion that the critical value of Rf is small.

The author knows of no measurements which are nearly precise enough to enable the conclusion to be checked. The work of Proudman (1953), which was referred to earlier, indicates a value around 0·25; but the conditions of the measurements were very different from the ideal considered in the theory and the discrepancy is not disturbing.

Even in neutral conditions it is known only that K_H/K_M is roughly unity. The work of Rider (1954) suggests a value about 1·3, while that of Swinbank (1955) suggests about 0·7. Cramer & Record (1953) found that K_M was often less than K_H, in agreement with Rider, but their results were definitely affected by the trajectory of the air in which the measurements were made. If we take $T_2/T_3 = 6$, $\overline{w^2}/u_*^2 = 1{\cdot}6$, and again $(\overline{u^2}+\overline{v^2}+\overline{w^2})/\overline{w^2} = 5{\cdot}5$, in neutral conditions $(Rf = 0)$, (20) gives $K_H/K_M = 1{\cdot}2$. That it should be necessary to take quite such a high value of T_2/T_3 is perhaps a little surprising, but quite unavoidable. It seems likely that with these values for the constants, (20) will give a good representation of K_H/K_M while Rf is small.

With increasing instability the ratios of the intensities will begin to change. At the extreme, in free convection, the scale of the motion is limited to be proportional to the height; but the turbulent energy is produced entirely by the buoyancy forces, and so, being independent of the shear-stress, must on dimensional grounds increase as $Z^{2/3}$. The equation for K_H/K_M now degenerates into one for K_H alone, namely,

$$K_H = -\frac{\bar{\rho}(\overline{u^2}+\overline{v^2}+\overline{w^2})\overline{w^2}[1+T_1(\overline{u^2}+\overline{v^2}+\overline{w^2})/T_2\overline{w^2}]}{2g\overline{\rho' w}T_2/T_3}. \tag{21}$$

But by (9),

$$K_H = H^{*2/3}|Z|^{1/3}u_*z = H^{*2/3}(g\overline{\rho' w}/\bar{\rho})^{1/3}z^{4/3},$$

so

$$(zg\overline{\rho' w}/\bar{\rho})^{4/3} = \frac{H^{*2/3}(\overline{u^2}+\overline{v^2}+\overline{w^2})\overline{w^2}[1+T_1(\overline{u^2}+\overline{v^2}+\overline{w^2})/T_2\overline{w^2}]}{2T_2/T_3}, \tag{22}$$

which connects the turbulent energy with the heat flux in free convection. We do not know the value of $(\overline{u^2}+\overline{v^2}+\overline{w^2})/\overline{w^2}$ and the ratio of the T's in these conditions. If (22) is considered as an equation for $\overline{u^2}+\overline{v^2}+\overline{w^2}$, it is not very sensitive to the former and we may take a value of 3 as being reasonable; in the absence of any evidence that the ratios of the T's differ from their values in neutral conditions, we can only make a provisional guess that they are the same. If so, we obtain

$$\overline{u^2}+\overline{v^2}+\overline{w^2} = 3(zg\overline{\rho' w}/\bar{\rho})^{2/3}. \tag{23}$$

Now let us turn to the second of the equations derived from (17) to (19), that for T_2, which is just a restatement of (18). T_2 has little direct interest since it is not measurable, but on account of our assertion that it is determined by typical length and velocity scales of the turbulence, it can to a limited extent help us to form a physical picture. $(\overline{u^2}+\overline{v^2}+\overline{w^2})^{1/2}$ is the obvious choice for the velocity scale, and so the equation for T_2 can be used to determine a length L_M defined as $T_2(\overline{u^2}+\overline{v^2}+\overline{w^2})^{1/2}$. From (18) and (6) we obtain

$$L_M = (\overline{u^2}+\overline{v^2}+\overline{w^2})^{3/2}u_*^{-3}LRf/(1-Rf). \tag{24}$$

This implies that L_M is proportional to z in neutral and unstable conditions and to L in very stable conditions, as was perhaps to be expected. The approximate numerical values are as follows: when $Rf = 0$, $L_M = 10z$; in free convection, $L_M = 5z$; in very stable conditions, $L_M = 5L$. It is of course only the relative sizes of L_M in the three states that is significant. It is a little surprising that its value in free convection is only half that in neutral conditions. This result may be partly spurious and due to an incorrect numerical constant in (23), but it is hardly possible that that is the whole cause of the difference. It is more likely that the effect is in some way connected with the fact that in free convection the turbulent energy is more nearly isotropically distributed than in neutral conditions.

The remaining equation to be discussed is that for $\overline{\rho'^2}$,

$$\frac{(\overline{w\rho'})^2}{\overline{\rho'^2}\,\overline{w^2}} = \frac{1-Rf[1+T_1(\overline{u^2}+\overline{v^2}+\overline{w^2})/T_2\overline{w^2}]}{2(T_1/T_3)(1-Rf)}. \tag{25}$$

If we keep the same numerical values as previously, this predicts that the correlation coefficient, $\overline{w\rho'}/(\overline{\rho'^2})^{1/2}(\overline{w^2})^{1/2}$, is about 0·6 in free convection; 0·3 in (almost) neutral conditions; and vanishes in very stable conditions. In principle these predictions can be tested easily, but again adequate measurements are lacking. The values given by Swinbank (1955) are very scattered, but indicate a value of 0·4 rather than 0·3 in nearly neutral conditions. The measurements do not show the form of the variation with stability beyond the sign of the general trend.

The simple physical explanation of the relative inefficiency of heat transfer in stable conditions expressed by (20) and (25) is that then a displaced fluid particle tends to return to its equilibrium level before it has had time to mix with its surroundings. A particle can transfer momentum during a brief excursion without mixing with its surroundings through the agency of pressure fluctuations, but in order to transfer heat it must mix.

Some idea of the vertical distance travelled by particles before either returning towards their equilibrium level or mixing can be obtained from the length $L_H = (\overline{\rho'^2})^{1/2}/d\rho/dz$. It can easily be shown that

$$L_H^2 = L_M^2\frac{T_3T_1}{2T_2^2}\frac{\overline{w^2}}{\overline{u^2}+\overline{v^2}+\overline{w^2}}\frac{1-Rf/Rf_{\text{crit}}}{1-Rf}, \tag{26}$$

which shows that in stable conditions L_H becomes much smaller than L_M. No precise interpretation of this can be given since the two scales have a different nature, but it presumably means that in stable conditions scales measured in the vertical direction are liable to be much smaller than those measured horizontally.

One last point is worth mentioning. In very stable conditions the last two terms of (19) become much greater than the first, and balance. Thus

$$\begin{aligned}\overline{w^2} &= \frac{\overline{\rho'^2}g}{\bar{\rho}d\bar{\rho}/dz} \\ &= \frac{gL_H^2}{\bar{\rho}}\frac{d\bar{\rho}}{dz}. \qquad (27)\end{aligned}$$

In other words, the scale L_H is just such that the work required to lift a particle a distance L_H from its equilibrium level is equal to its average kinetic energy.

7. Transport of inert pollutant

If the theory of the previous section is applied in a similar way to the transport of an inert pollutant (i.e. one that does not affect the density of the fluid), it is found that the *Austausch* coefficient for it is necessarily equal to K_H. The reason for this is that no mechanism is included in the theory which would enable the turbulence to destroy the correlation between density fluctuation and pollutant concentration. Thus the conclusion is one which may be seriously upset as the role of molecular processes in turbulent flow becomes better understood. Observational evidence based on direct measurements of the *Austausch* coefficients is at present conflicting; and it is likely that measurements of the correlation between density fluctuation and pollutant concentration would provide a simpler and more sensitive test.

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